

# Primordial Fluctuations of the metric in the Warm Inflation Scenario

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I consider a semiclassical expansion of the scalar field in the warm inflation scenario. I study the evolution for the fluctuations of the metric around the flat Friedmann - Robertson - Walker one. The formalism predicts that, in a power - law expansion of the universe, the fluctuations of the metric decreases with time.

## I. INTRODUCTION

The standard inflation scenario describes a quasi-de Sitter expansion in a supercooled scenario. This model separates expansion and reheating into two distinguished time periods [1]. This theory assumes a second order phase transition of the inflaton field, followed by a localized mechanism that rapidly distributes the vacuum energy into thermal energy. Reheating after inflation occurs due to particle production by the oscillating inflaton field.

Warm Inflation is a theory of the early universe that explains the abrupt expansion and the exchange of energy between the inflaton field and the thermal bath. This theory was proposed by A. Berera a few years ago [2], and generalized in another works [3]. Quantum to classical transition [4] and the power spectrum of the primordial fluctuations was studied [5] in the framework of a stochastic approach for the warm inflation scenario. In this thermal scenario the rapid cooling followed by rapid heating is replaced by a smoothed dissipative mechanism. The warm inflation formalism predicts energy fluctuations  $\frac{\delta\rho}{\rho}$  that decreases with time in a power - law expansion, when the universe grows very rapidly [4,5]. In this formalism a semiclassical expansion for the inflaton field  $\varphi$  was proposed  $\varphi(\vec{x}, t) = \phi_c(t) + \phi(\vec{x}, t)$ , where  $\vec{x}$  are the spatial coordinates,  $t$  is the time,  $\phi_c(t)$  the spatially homogeneous field and  $\phi(\vec{x}, t)$  are the quantum fluctuations. For consistency one requires  $\langle \varphi(\vec{x}, t) \rangle = \phi_c(t)$ , and  $\langle \phi(\vec{x}, t) \rangle = \langle \dot{\phi}(\vec{x}, t) \rangle = 0$ . In the warm inflation scenario the rapid expansion of the universe is produced in presence of a thermal component. The kinetic energy density ( $\rho_{kin} = \rho_r + \frac{1}{2}\dot{\varphi}^2$ ) must be small with respect to the vacuum energy density:  $\rho(\varphi) \sim \rho_m(\varphi) \sim V(\varphi) \gg \rho_{kin}$ , where  $V(\varphi)$  is the potential associated with the field of matter  $\varphi$ . Furthermore  $\rho_m$  and  $\rho_r = \frac{\tau(\varphi)}{8H(\varphi)}\dot{\varphi}^2$  are the matter and radiation energy densities. This scenario provides thermal fluctuations compatible with de COBE data [6] if the thermal equilibrium becomes near the minimum of the potential  $V(\varphi)$  [7]. Furthermore, particles are created during the expansion of the universe, and it is not necessary a further reheating era. The field  $\varphi$  interacts with other particles, which there are in the thermal bath at temperature  $T_r < T_{GUT} \sim 10^{15}$  GeV.

The Lagrangian density that describes the warm inflation scenario is

$$\mathcal{L}(\varphi, \varphi_{,\mu}) = -\sqrt{-g} \left[ \frac{R}{16\pi} + \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + V(\varphi) \right] + \mathcal{L}_{int}, \quad (1)$$

where  $R$  is the scalar curvature,  $g^{\mu\nu}$  the metric tensor,  $g$  is the metric and  $\mathcal{L}_{int}$  takes into account the interaction of the field  $\varphi$  with other fields of the thermal bath. In this work I consider a perturbed Friedmann - Robertson - Walker (FRW)

$$ds^2 = -dt^2 + a^2(t) [1 + h(\vec{x}, t)] d\vec{x}^2. \quad (2)$$

Here,  $a(t)$  is the scale factor and  $h(\vec{x}, t)$  denote the fluctuations around the FRW metric with zero global curvature ( $k = 0$ ). The perturbations  $h(\vec{x}, t)$  are assumed to be small. Expanding  $H[\varphi(\vec{x}, t)]$  around  $\phi_c$ , one obtains  $H(\varphi) \simeq H_c(\phi_c) + H'(\phi_c) \phi(\vec{x}, t)$ , at first order on  $\phi$ . The perturbations of the metric, for small  $\phi$ , are

$$h(\vec{x}, t) \simeq 2 \int H'_c(\phi_c) \phi(\vec{x}, t) dt. \quad (3)$$

The equation of motion for the scalar field  $\varphi$  is

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$$\ddot{\varphi} - \frac{1}{a^2} \nabla^2 \varphi + [3H(\varphi) + \tau(\varphi)] \dot{\varphi} + V'(\varphi) = 0, \quad (4)$$

where  $\tau(\varphi) \dot{\varphi}$  describes the dissipation due to the interaction of the field  $\varphi$  with the fields of the thermal bath. We write the semiclassical Friedmann equation for a globally flat FRW metric, which describes a globally isotropic and homogeneous universe:  $\langle H^2(\varphi) \rangle = \langle \frac{8\pi}{3} G (\rho_m + \rho_r) \rangle$ . Here  $G = M_p^{-2}$  is the gravitational constant and  $M_p$  the Planckian mass. The matter and radiation energy densities are  $\rho_m(\varphi) = \frac{\dot{\varphi}^2}{2} + \frac{1}{a^2} (\vec{\nabla} \varphi)^2 + V(\varphi)$  and  $\rho_r(\varphi) = \frac{\tau(\varphi)}{8H(\varphi)} \dot{\varphi}^2$ .

## II. THE CLASSICAL FIELD $\phi_c$

As in previous works [5,7], I consider the following relation between the friction parameter  $\tau_c$ , and the Hubble one

$$\tau_c(\phi_c) = \gamma H_c(\phi_c), \quad (5)$$

where  $\gamma$  is a dimensionless constant. The classical field is defined as a solution of the equation of motion

$$\ddot{\phi}_c + [3H_c(\phi_c) + \tau_c(\phi_c)] \dot{\phi}_c + V'(\phi_c) = 0, \quad (6)$$

on the unperturbed FRW metric:  $\langle ds^2 \rangle = \langle -dt^2 + a^2(t) [1 + h(\vec{x}, t)] d\vec{x}^2 \rangle = -dt^2 + a^2(t) d\vec{x}^2$ .

The classical Hubble parameter is

$$H_c^2(\phi_c) = \frac{4\pi}{3M_p^2} \left[ \left( 1 + \frac{\tau_c}{4H_c} \right) \dot{\phi}_c^2 + 2V(\phi_c) \right], \quad (7)$$

and thus, the scalar potential is

$$V(\phi_c) = \frac{3M_p^2}{8\pi} \left[ H_c^2(\phi_c) - \frac{M_p^2}{12\pi} (H'_c)^2 \left( 1 + \frac{\tau_c}{4H_c} \right) \left( 1 + \frac{\tau_c}{3H_c} \right)^{-2} \right], \quad (8)$$

where one assumes that  $H(\varphi) = H(\phi_c) \equiv H_c$  and  $\tau(\varphi) = \tau(\phi_c) \equiv \tau_c$ . The eq. (8) is obtained using the following equations that describe the dynamics of the classical field and the Hubble parameter:  $\dot{\phi}_c = -\frac{M_p^2}{4\pi} H'_c \left( 1 + \frac{\tau_c}{3H_c} \right)^{-1}$ , and  $\dot{H}_c = -\frac{M_p^2}{4\pi} (H'_c)^2 \left( 1 + \frac{\tau_c}{3H_c} \right)^{-1}$ , where the prime denote the derivative with respect to the field. With the equation for  $\dot{\phi}_c$ , one obtains the expression for the radiation energy density

$$\rho_r(\phi_c) = \frac{\tau_c}{8H_c} \left( \frac{M_p^2}{4\pi} \right)^2 (H'_c)^2 \left( 1 + \frac{\tau_c}{3H_c} \right)^{-2}. \quad (9)$$

When the thermal equilibrium holds, the temperature of the bath is  $\langle T_r \rangle \propto \left( \frac{\tau_c(\phi_c)}{8H_c(\phi_c)} \dot{\phi}_c^2 \right)^{1/4} t_{\gg 1}$ . In this formalism, the expansion of the universe and the interaction of the inflaton with the bath are produced by the classical field  $\phi_c$ .

## III. THE QUANTUM FLUCTUATIONS

For simplicity, I consider the quantum fluctuations on the expectation value of the metric

$$\langle ds^2 \rangle = -dt^2 + a(t) d\vec{x}^2. \quad (10)$$

However, a consistent treatment must consider the interaction of the quantum fluctuations with the metric. Here, the simplification  $\langle H[\varphi(\vec{x}, t)] \rangle = H_c(\phi_c)$  will be assumed. The equation of motion for the quantum fluctuations [with the simplification (10)], is [5,7]

$$\ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + [3H_c + \tau_c] \dot{\phi} + V''(\phi_c) \phi = 0. \quad (11)$$

The structure of this equation can be simplified with the mapp  $\chi = e^{3/2 \int (H_c + \tau_c/3) dt} \phi$ , and one obtains

$$\ddot{\chi} - a^{-2} \nabla^2 \chi - \frac{k_o^2}{a^2} \chi = 0, \quad (12)$$

where  $k_o^2(t) = a^2 \left[ \frac{9}{4} (H_c + \frac{\tau_c}{3})^2 - V''(\phi_c) + \frac{3}{2} \left( \dot{H}_c + \frac{\dot{\tau}_c}{3} \right) \right]$ . The field that describes the quantum fluctuations can be written as a Fourier expansion of the modes  $\xi_k(t) e^{i\vec{k} \cdot \vec{x}}$

$$\chi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \left[ a_k e^{i\vec{k} \cdot \vec{x}} \xi_k(t) + h.c. \right]. \quad (13)$$

Since in the metric (10) there are not taken into account the quantum fluctuations, the field (13) can be consider as free. However, in a more consistent formalism must be considered the interaction of the quantum fluctuations with the fluctuations of the metric. I denote with  $a_k^\dagger$  and  $a_k$ , the creation and annihilation operators. These operators satisfy the commutation relations  $[a_k, a_{k'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}')$  and  $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$ . The commutation relation between the operators  $\dot{\chi}$  and  $\chi$  is

$$[\chi(\vec{x}, t), \dot{\chi}(\vec{x}, t)] = i\delta(\vec{x} - \vec{x}'), \quad (14)$$

which is satisfied for  $\xi_k \dot{\xi}_k^* - \dot{\xi}_k \xi_k^* = i$ .

I am interested in the study of the universe on a scale greater than the observable universe. Thus, I consider the field  $\chi$ , but taking into account only the modes with wavelength of size  $l \geq \frac{1}{\epsilon k_o}$ , where  $\epsilon \ll 1$  is a dimensionless constant. This field is given by

$$\chi_{cg}(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \theta(\epsilon k_o - k) \left[ a_k e^{i\vec{k} \cdot \vec{x}} \xi_k + h.c. \right]. \quad (15)$$

The coarse - grained field (15) satisfy the following stochastic equation

$$\ddot{\chi}_{cg} - \frac{k_o^2}{a^2} \chi_{cg} = \epsilon \left( \frac{d}{dt} \left( k_o \eta \right) + 2 \dot{k}_o \kappa \right), \quad (16)$$

with the noises

$$\begin{aligned} \eta(\vec{x}, t) &= \frac{1}{(2\pi)^{3/2}} \int d^3 k \delta(\epsilon k_o - k) \left[ a_k e^{i\vec{k} \cdot \vec{x}} \xi_k + h.c. \right], \\ \kappa(\vec{x}, t) &= \frac{1}{(2\pi)^{3/2}} \int d^3 k \delta(\epsilon k_o - k) \left[ a_k e^{i\vec{k} \cdot \vec{x}} \dot{\xi}_k + h.c. \right]. \end{aligned}$$

The equation (16) is an second order operatorial stochastic equation, and also can be obtained in the standard inflation formalism (for  $V(\varphi) = 0$ ) [8].

Complex to real transition of the modes  $\xi_k$  was studied in a previous article [7]. When this transition occurs one obtains  $\xi_k(t) \xi_k^*(t') \simeq \xi_k(t) \xi(t')$ , and the commutators  $[\chi_{cg}, \eta]$ ,  $[\chi_{cg}, \kappa]$ ,  $[\eta, \kappa]$ , and (14) are null. Hence, the self - correlation of the coarse - grained field  $\chi_{cg}$  becomes

$$\langle \chi_{cg}(t) \chi_{cg}(t') \rangle = \frac{1}{(2\pi)^{3/2}} \int_{\epsilon k_o(t)}^{\epsilon k_o(t')} d^3 k \xi_k(t) \xi_k(t'). \quad (17)$$

The radiation energy fluctuations are  $\delta \rho_r(t) = \left| 2 \left( \frac{\gamma}{8} \right) \left( 1 + \frac{\gamma}{3} \right)^{-2} H'_c H''_c \right| \langle \phi_{cg}^2 \rangle^{1/2}$ , and the radiation energy density is  $\rho_r(t) = \left( \frac{\gamma}{8} \right) \left( 1 + \frac{\gamma}{3} \right)^{-2} (H'_c)^2$ . When the radiation energy fluctuations  $\frac{\delta \rho_r}{\rho_r}$ , decreases with time, the thermal equilibrium holds for sufficiently large times. In this case, the Fourier transform of (17) gives the spectral density for the quantum fluctuations in the infrared sector

$$S[\chi_{cg}; \omega_k] = \frac{1}{\pi} \int_0^\infty dt'' \cos[\omega t''] \int_{\epsilon k_o(t)}^{\epsilon k_o(t+t'')} dk k^2 \xi_k(t) \xi_k(t+t''). \quad (18)$$

Here,  $\omega_k$  is the frequency of oscillation for each mode with wavenumber  $k$ . In a semiclassical representation to the warm inflation scenario, the fluctuations of the metric are dues to the quantum fluctuations  $\phi_{cg} = e^{-3/2 \int (H_c + \tau_c/3) dt} \chi_{cg}$ , on

a scale much greater than the scale of the observable universe. The perturbed metric for the FRW universe is given by eq. (2), with  $h(\vec{x}, t)$  given by eq. (3). The temporal evolution of  $h(\vec{x}, t)$  is given by

$$\langle h^2(t) \rangle^{1/2} \simeq a^{-2} \left[ 2 \int H'(\phi_c) \langle \phi_{cg}^2(\vec{x}, t) \rangle^{1/2} dt \right]. \quad (19)$$

Quantum to classical transition of the perturbations of the metric holds due to the quantum to classical transition of the coarse - grained quantum field  $\phi_{cg}$ . It is produced when all of the modes of  $\chi_{cg}$  become real, i.e. [7]:  $\frac{1}{N(t)} \sum_{k=0}^{k=\epsilon k_o} \alpha_k(t) \ll 1$ . Here  $N(t)$  is the time dependent number of degrees of freedom of the infrared sector, which increases with time. The function  $\alpha_k(t)$  was defined in a previous work [5], and is:  $\alpha_k = \left| \frac{v_k(t)}{u_k(t)} \right|$ , for  $\xi_k(t) = u_k(t) + i v_k(t)$ . This effect is due to the time evolution of the superhorizon with size  $l(t) \geq \frac{1}{\epsilon k_o}$ . Thus, quantum to classical transition of the coarse - grained field occurs when  $\alpha_{k=\epsilon k_o} \rightarrow 0$ .

#### IV. AN EXAMPLE: FLUCTUATIONS OF THE METRIC IN POWER - LAW INFLATION

In this section I estimate the fluctuations of the metric in the example for power - law inflation. Here, the scale factor and the Hubble parameter are  $a(t) = H_o^{-1}(t/t_o)^p$ , and  $H_c(t) = \frac{p}{t}$ , respectively. The temporal evolution of the scalar field is  $\phi_c(t) = \phi_o - m \ln \left[ \frac{H_o}{p} t \right]$ . Hence, the scalar potential  $V(\phi_c)$  is given by

$$V(\phi_c) = \frac{3M_p^2 H_o^2}{8\pi} e^{2\phi_c/m} \left[ 1 - \frac{M_p^2}{12\pi m^2} \left( 1 + \frac{\gamma}{4} \right) \left( 1 + \frac{\gamma}{3} \right)^{-2} \right]. \quad (20)$$

The temporal evolution of the quantum fluctuations is

$$\langle \phi_{cg}^2(t) \rangle^{1/2} \Big|_{t \gg 1} \propto t^{2\nu p(1-\nu/p-p/\nu)+3/2-\gamma/2}, \quad (21)$$

with  $\nu = \frac{\sqrt{1+9p^2(1+\gamma)^2-6p[(1+\gamma)+1]+9(1+\frac{\gamma}{4})(1+\frac{\gamma}{3})^{-2}}}{2(p-1)}$ . These fluctuations decrease with time when the universe grows very rapidly (for  $p \gg 1$ ). So, the fluctuations of the metric are given (as  $H'_c(t) \sim t^{-1}$ ) by

$$\langle h^2(t) \rangle^{1/2} \sim t^{2\nu p(1-\nu/p-p/\nu)+3/2-\gamma/2}, \quad (22)$$

which, since  $\langle \phi_{cg}^2 \rangle^{1/2}$ , decreases with time for  $p \gg 1$ . The power spectral density for the matter field fluctuations is  $S[\chi_{cg}, \omega_k = \frac{2k}{t^p}] \propto |\omega_k|^n$ , with  $n = -[4\nu(p-1) - 2p + 4]$ . The fluctuations of radiation energy density are

$$\frac{\delta \rho_r}{\rho_r} \Big|_{t \gg 1} \sim t^{2\nu p(1-\nu/p-(5/4)p/\nu)+1/2-\gamma/2}, \quad (23)$$

which decreases for  $p$  sufficiently large. Thus, for  $p \gg 1$  the thermal equilibrium holds due to  $\frac{d[\delta \rho_r / \rho_r]}{dt} < 0$ .

#### V. FINAL COMMENTS

In this work was studied the evolution for the fluctuations of the metric with a semiclassical representation of the inflaton field  $\varphi$ , in the warm inflation scenario. The mean temperature is smaller than the GUT temperature (i.e.,  $\langle T_r \rangle < T_{GUT} \simeq 10^{15}$  GeV). In this theory the classical matter field lead to the expansion of the universe, while the fluctuations of the matter field generate the inhomogeneities of the metric, and thus local curvature of the spacetime. However the expectation value of the curvature is zero, in consistency with a globally flat perturbed Friedmann - Robertson - Walker metric here considered. In this framework the quantum fluctuations averaged over a scale much bigger than the observable universe are responsible (not only of the radiation energy density fluctuations, thermal fluctuations and matter energy density fluctuations - these topics were studied in another previous works [4,5,7]) for the fluctuations of the metric. This calculation was done with the assumption that the quantum fluctuations are very

small. Thus, was possible to expand the coarse - grained field as free. In a more appropriate treatment for the coarse - grained field, it must be consider as interactuant with the metric.

In the example here considered I observe that the fluctuations of the metric decreases with time, in a power - law expansion when the expansion is very rapid. Furthermore, the radiation energy fluctuations decreases with time for a sufficiently large rate of expansion of the universe. Hence, when the universe expands very rapidly, the thermal equilibrium holds.

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